

LAUDAL'S LEMMA IN POSITIVE CHARACTERISTIC

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ABSTRACT. Laudal's Lemma states that if C is a curve of degree $d > s^2 + 1$ in \mathbb{P}^3 over an algebraically closed field of characteristic 0 such that its plane section is contained in an irreducible curve of degree s , then C lies on a surface of degree s . We show that the same result does not hold in positive characteristic and we find different bounds $d > f(s)$ which ensure that C is contained in a surface of degree s .

1. INTRODUCTION

Let C be a curve in \mathbb{P}_k^3 , being k an algebraically closed field. Let Γ be the generic plane section of C . In this paper we study the problem of finding bounds on the degree of C in such a way that, if Γ is contained in a plane curve of degree s , then C is contained in a surface of the same degree. In the case that $\text{char } k = 0$ the following result has been proved:

Theorem 1.1 (Laudal's Lemma, [11, Corollary, p.147],[6]). *If Γ is contained in an integral plane curve of degree s and $\deg C > s^2 + 1$, then C is contained in a surface of degree s .*

The bound on the degree of the curve found in this result is sharp. Indeed, there are examples of curve of degree $s^2 + 1$ whose the generic plane section is contained in an irreducible plane curve of degree s and that are not contained in any surface of degree s (see [7], [6] and [13, Proposition 1]).

In this paper, following the proof of Gruson and Peskine of Laudal's Lemma in [6], we prove an analogous result in the case that the field k has positive characteristic:

Theorem 1.2. *Let $C \subset \mathbb{P}^3$ be a non degenerate reduced curve of degree d in characteristic $p > 0$. Suppose that Γ is contained in an integral plane curve of degree s . Then C is contained in a surface of degree s , if one of the following conditions is satisfied:*

- (1) C is connected, $p \geq s$ and $d > s^2 + 1$;
- (2) C is connected, $p < s$ and $d > s^2 + p^{2n}$, with $p^n < s \leq p^{n+1}$; in particular this holds if $d > 2s^2 - 2s + 1$;
- (3) $p > s$ and $d > s^2 + 1$;
- (4) $p \leq s$ and $d > s^2 + p^{2n}$, with $p^n \leq s < p^{n+1}$. In particular this holds if $d > 2s^2$.

Let us make a note about terminology. Given the incidence variety $T = \{([H], P) \in \check{\mathbb{P}}^3 \times \mathbb{P}^3 \mid P \in C \cap H\}$ associated to C , the fibre of the projection $T \rightarrow \check{\mathbb{P}}^3$ over the generic point $\eta \in \check{\mathbb{P}}^3$ is the generic plane section Γ . In particular we consider the open subset $U \subset \check{\mathbb{P}}^3$ such that any $[H] \in U$ corresponds to a plane $H \subset \mathbb{P}^3$ that C meets transversally and such planes are generic for the curve C .

Let us give now a sketch of the proof of Theorem 1.2, given in Section 4. We follow the idea of the proof of Theorem 1.1 given by Gruson and Peskine in [6]. So we take $S \subset \mathbb{P}^3 \times \mathbb{P}^3$ containing T such that the fibre over η is an integral plane curve of degree s containing Γ . Then we suppose that $h^0(\mathcal{I}_C(s)) = 0$ and, using Theorem 3.3, that is the main result of Section 3, we factor the projection $S \rightarrow \mathbb{P}^3$ through a generically smooth morphism $S_r \rightarrow \mathbb{P}^3$, with $S_r = S \times_{\mathbb{P}^3, F^r} \mathbb{P}^3$ and F^r some r -th power of the absolute Frobenius of \mathbb{P}^3 . Then, proceeding as in [6], we arrive to the inequality $d \leq s^2 + p^{2r}$. Remarking that it must be $h^1(\mathcal{I}_C(s - p^r)) \neq 0$ we find the desired inequalities.

Looking at the proof we see that the assumption that C is reduced is required to find a suitable bound to the power p^r . Moreover, in the case that C is connected this bound is sharp, as we will see in Example 5.3. Indeed, generalizing the example given in [6] and [13, Proposition 1] to prove that the bound in Theorem 1.1 is sharp, we consider the sheaf $\mathcal{E} = F^{n*}(\mathcal{E}_0)$, with \mathcal{E}_0 null-correlation bundle and F absolute Frobenius on \mathbb{P}^3 . Then the zero locus of a generic global section of $\mathcal{E}(s)$, for $s > p^n$, is an integral curve of degree $s^2 + p^{2n}$ not lying on any surface of degree s such that its generic plane section is contained in an integral plane curve of degree s .

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2. THE FROBENIUS MORPHISM

First let us recall the definition of the relative Frobenius morphism (we follow Ein's notation in [3]):

Definition 2.1. The absolute Frobenius morphism of a scheme X of characteristic $p > 0$ is $F_X: X \rightarrow X$, where F_X is the identity as a map of topological spaces and on each U open set $F_X^\#: \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(U)$ is given by $f \mapsto f^p$ for each $f \in \mathcal{O}_X(U)$. Given $X \rightarrow S$ for some scheme S and $X^{p/S} = X \times_{S, F_S} S$, the absolute Frobenius morphisms on X and S induce a morphism $F_{X/S}: X \rightarrow X^{p/S}$, called the Frobenius morphism of X relative to S .

Let now $r \in \mathbb{N}$ and $n \in \mathbb{Z}$ be integers. Let $F: \mathbb{P}^3 \rightarrow \mathbb{P}^3$ be the absolute Frobenius and let us consider the sheaf $\mathcal{F} = (F^r)^*(\Omega_{\mathbb{P}^3})$. The following result will be needed later:

Lemma 2.2.

- (i) $h^0(\mathcal{F}(2p^r)) = 6$,
- (ii) $h^0(\mathcal{F}(n)) \neq 0$ if and only if $n \geq 2p^r$,
- (iii) $h^2(\mathcal{F}(n)) = 0$ for every $n \in \mathbb{Z}$,

Proof. First let us make some remarks. The sheaf $\Omega_{\mathbb{P}^3}$ is determined by the Euler sequence $0 \rightarrow \Omega_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0$, which is part of the Koszul complex $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-3) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 6}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0$. So \mathcal{F} , by the flatness of the absolute Frobenius, is determined by the exact sequence:

$$(1) \quad 0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-p^r) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0,$$

which is part of the following long exact sequence:

$$(2) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-4p^r) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-3p^r) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 6}(-2p^r) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-p^r) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0.$$

(iii) follows immediately from (1). Now we prove (i) and (ii). Considered the cokernel \mathcal{G} of the first nonzero map in (2):

$$(3) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-4p^r) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-3p^r) \rightarrow \mathcal{G} \rightarrow 0$$

\mathcal{F} and \mathcal{G} are related by the exact sequence:

$$(4) \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 6}(-2p^r) \rightarrow \mathcal{F} \rightarrow 0.$$

Since \mathcal{F} and \mathcal{G} are vector bundles and $\mathcal{F}^\vee \cong \mathcal{G}(4p^r)$, then they are reflexive and $\mathcal{G}^\vee \cong \mathcal{F}(4p^r)$. So by (3) and (4):

$$h^0(\mathcal{F}(n)) = 6h^0(\mathcal{O}_{\mathbb{P}^3}(n - 2p^r)) - 4h^0(\mathcal{O}_{\mathbb{P}^3}(n - 3p^r)) + h^0(\mathcal{O}_{\mathbb{P}^3}(n - 4p^r)).$$

From this we get (i) and (ii). \blacksquare

In the notation of Lemma 2.2, let us consider the sheaf $\mathcal{K} = \mathcal{F}(p^r)|_H$, restriction of $\mathcal{F}(p^r)$ to a plane H in \mathbb{P}^3 .

Lemma 2.3. *For every $m \in \mathbb{Z}$:*

$$h^0(\mathcal{K}(m)) = h^0(\mathcal{O}_H(m)) + 3h^0(\mathcal{O}_H(m - p^r)) - h^0(\mathcal{O}_H(m - 2p^r)).$$

Proof. Let us make the position $\mathcal{F}_H = (F^r)^*(\Omega_H)$. We can construct a surjective morphism of sheaves $\varphi : \mathcal{O}_H^{\oplus 4} \rightarrow \mathcal{O}_H^{\oplus 3}$ in such a way that we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{O}_H^{\oplus 4} & \longrightarrow & \mathcal{O}_H(p^r) \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \downarrow \text{id} \\ 0 & \longrightarrow & \mathcal{F}_H(p^r) & \longrightarrow & \mathcal{O}_H^{\oplus 3} & \longrightarrow & \mathcal{O}_H(p^r) \longrightarrow 0. \end{array}$$

Indeed, if $k[x_0, x_1, x_2, x_3]$ is the coordinate ring associated to \mathbb{P}^3 and H has equation $x_3 = \sum_{i=0}^2 a_i x_i$, then we can define φ as given by $\mathcal{O}_H^{\oplus 4} \ni (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_0 + a_0 p^r \sigma_3, \sigma_1 + a_1 p^r \sigma_3, \sigma_2 + a_2 p^r \sigma_3) \in \mathcal{O}_H^{\oplus 3}$.

So by the snake lemma and by the fact that $\text{Ker } \varphi \cong \mathcal{O}_H$ we find the exact sequence:

$$(5) \quad 0 \rightarrow \mathcal{O}_H \rightarrow \mathcal{K} \rightarrow \mathcal{F}_H(p^r) \rightarrow 0.$$

Proceeding as in Lemma 2.2 we see that $\mathcal{F}_H(p^r)$ comes from the Koszul complex $0 \rightarrow \mathcal{O}_H(-2p^r) \rightarrow \mathcal{O}_H^{\oplus 3}(-p^r) \rightarrow \mathcal{O}_H^{\oplus 3} \rightarrow \mathcal{O}_H(p^r) \rightarrow 0$, which implies that $\mathcal{F}_H = \mathcal{F}_H^\vee(-3p^r)$. Now:

$$(6) \quad \begin{aligned} h^0(\mathcal{F}_H(p^r + m)) &= h^0(\mathcal{F}_H^\vee(m - 2p^r)) = h^0((\mathcal{F}_H(2p^r - m))^\vee) = \\ &= h^2(\mathcal{F}_H(2p^r - m) \otimes \mathcal{O}_H(-3)) = h^2(\mathcal{F}_H(2p^r - m - 3)). \end{aligned}$$

From $0 \rightarrow \mathcal{F}_H(2p^r - m - 3) \rightarrow \mathcal{O}_H^{\oplus 3}(p^r - m - 3) \rightarrow \mathcal{O}_H(2p^r - m - 3) \rightarrow 0$ we see that $h^2(\mathcal{F}_H(2p^r - m - 3)) = -h^0(\mathcal{O}_H(m - 2p^r)) + 3h^0(\mathcal{O}_H(m - p^r))$, that, together with (5) and (6), leads us to the conclusion. \blacksquare

3. INCIDENCE VARIETIES IN CHARACTERISTIC p

Let us consider the bi-projective space $\check{\mathbb{P}}^3 \times \mathbb{P}^3$ and let $r \in \mathbb{N}$ be a non negative integer. Let $k[\underline{t}]$ and $k[\underline{x}]$ be the coordinate rings for $\check{\mathbb{P}}^3$ and \mathbb{P}^3 , respectively. Let M_r be the hypersurface of equation:

$$h_r := \sum_{i=0}^3 t_i x_i^{p^r} = 0.$$

First we need the following result:

Lemma 3.1. *Let $q \in k[\underline{t}, \underline{x}]$ be a homogeneous polynomial of bi-degree (α, s) such that:*

$$(7) \quad x_i^{p^r} \frac{\partial q}{\partial t_j} - x_j^{p^r} \frac{\partial q}{\partial t_i} \in (h_r) \quad \forall i, j$$

and $q \notin (h_r)$. Then there exists $q' = q + h_r m$ bi-homogeneous of bi-degree (α, s) such that:

$$\frac{\partial q'}{\partial t_i} = 0 \quad \forall i.$$

Since the proof of this lemma requires some computations, we leave it to the end of this section. Let us now remark that in the case $r = 0$ M_r is usual incidence variety M of equation $\sum t_i x_i = 0$. If $r \geq 1$, M_r is determined by the following fibred product:

$$(8) \quad \begin{array}{ccccc} M & & & & \\ & \searrow^{(F_M)^r} & & & \\ & & M_r & \xrightarrow{\pi} & M \\ & \swarrow_p & \downarrow p_{M_r} & & \downarrow p \\ & & \mathbb{P}^3 & \xrightarrow{F^r} & \mathbb{P}^3 \end{array}$$

where $F: \mathbb{P}^3 \rightarrow \mathbb{P}^3$ is the absolute Frobenius. Moreover $M = \mathbb{P}(\Theta_{\mathbb{P}^3}(-1))$ and so by [3, Lemma 1.5] we get $M_r = \mathbb{P}(F^{r*}(\Theta_{\mathbb{P}^3}(-1)))$. By [8, Ch.II, ex. 7.8] this implies:

$$(9) \quad \text{Pic}(M_r) = \mathbb{Z} \times \mathbb{Z}$$

for any $r \geq 0$.

Let us consider an integral hypersurface $V \subset M_r$ and let us suppose that the projection $\pi: V \rightarrow \mathbb{P}^3$ is dominant. Using the previous lemma we prove the following result:

Proposition 3.2. *If π is not generically smooth, then there exists $s \geq 1$, such that $V \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ is the complete intersection determined by $g = h_r = 0$, for some $g \in k[\underline{t}^{p^s}, \underline{x}]$.*

Proof. Since $M_r \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ is a hypersurface of bi-degree $(1, p^r)$, the structure sheaf \mathcal{O}_{M_r} is given by $0 \rightarrow \mathcal{O}_{\check{\mathbb{P}}^3 \times \mathbb{P}^3}(-1, -p^r) \rightarrow \mathcal{O}_{\check{\mathbb{P}}^3 \times \mathbb{P}^3} \rightarrow \mathcal{O}_{M_r} \rightarrow 0$. By the Künneth formula ([12, Ch. VI, Corollary 8.13]) $h^1(\mathcal{O}_{\check{\mathbb{P}}^3 \times \mathbb{P}^3}(m, n)) = 0$ for every $m, n \in \mathbb{Z}$, so that the morphism $H^0(\mathcal{O}_{\check{\mathbb{P}}^3 \times \mathbb{P}^3}(m, n)) \rightarrow H^0(\mathcal{O}_{M_r}(m, n))$ is surjective for every $m,$

$n \in \mathbb{Z}$. Together with (9) this implies that $V \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ is a complete intersection given by $g = h_r = 0$ for some $g \in k[\underline{t}, \underline{x}]$ bi-homogeneous of bi-degree (m, n) for some $m, n \in \mathbb{N}$.

Let $P_0 = (\underline{a}, \underline{b}) \in V$ be a regular point. By hypothesis the map on the projective tangent spaces T_{V, P_0} and $T_{\mathbb{P}^3, \pi(P_0)}$ is not surjective. The projective tangent space T_{V, P_0} at $P_0 \in V$ is given by the equations:

$$\sum_{i=0}^3 \frac{\partial g}{\partial x_i}(P_0)x_i + \sum_{i=0}^3 \frac{\partial g}{\partial t_i}(P_0)t_i = \sum_{i=0}^3 (a_i x_i + b_i t_i) = 0$$

if $r = 0$ and by the equations:

$$\sum_{i=0}^3 \frac{\partial g}{\partial x_i}(P_0)x_i + \sum_{i=0}^3 \frac{\partial g}{\partial t_i}(P_0)t_i = \sum_{i=0}^3 b_i^{p^r} t_i = 0$$

if $r \geq 1$. In both cases the projection on $T_{\mathbb{P}^3, \pi(P_0)}$ is not surjective if and only if there exists $\lambda \in k$ such that:

$$\frac{\partial g}{\partial t_i}(P_0) = \lambda b_i^{p^r} \quad \forall i = 0, \dots, 3.$$

So in such a situation:

$$b_i^{p^r} \frac{\partial g}{\partial t_j}(P_0) - b_j^{p^r} \frac{\partial g}{\partial t_i}(P_0) = 0 \quad \forall i, j.$$

This means that for every i, j the hypersurface $V_{ij} \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ given by $x_i^{p^r} \frac{\partial g}{\partial t_j} - x_j^{p^r} \frac{\partial g}{\partial t_i} = 0$ contains $\text{Reg}(V)$, the open subset of the regular points of V . So $V_{ij} \supset V$ for all i, j , which means that:

$$x_i^{p^r} \frac{\partial g}{\partial t_j} - x_j^{p^r} \frac{\partial g}{\partial t_i} \in (g, h_r) \quad \forall i, j.$$

If $x_i^{p^r} \frac{\partial g}{\partial t_j} - x_j^{p^r} \frac{\partial g}{\partial t_i}$ is a nonzero polynomial, then it is a bi-homogeneous polynomial of bi-degree $(m-1, n+p^r)$. Since g is bi-homogeneous of bi-degree (m, n) , then

$$x_i^{p^r} \frac{\partial g}{\partial t_j} - x_j^{p^r} \frac{\partial g}{\partial t_i} \in (h_r) \quad \forall i, j.$$

Applying Lemma 3.1 we see that there exists $m \in k[\underline{t}, \underline{x}]$ such that, given $g' = g + mh_r$, we have $\partial g' / \partial t_i = 0$ for every i . So by replacing g by g' we can suppose that $g \in k[\underline{t}^{p^s}, \underline{x}]$, for some $s \geq 1$. \blacksquare

Now we can prove the main result of this section:

Theorem 3.3. *Let $V \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ be an integral hypersurface in M such that the projection $\pi: V \rightarrow \mathbb{P}^3$ is dominant and not generically smooth. Then there exist $r \geq 1$, and $V_r \subset M_r$ integral hypersurface such that π can be factored in the following way:*

$$\begin{array}{ccc} V & \xrightarrow{\pi} & \mathbb{P}^3 \\ & \searrow F_r & \nearrow \pi_r \\ & V_r & \end{array}$$

where the projection π_r is dominant and generically smooth and F_r is induced by the commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{F_r} & V_r \\ j \downarrow & & \downarrow i \\ M & \xrightarrow{F_{M_r}} & M_r. \end{array}$$

Proof. First note that by hypothesis and by Proposition 3.2 it follows that $V \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ is the complete intersection determined by $h = 0$ and $q = 0$, for some $q \in k[\underline{t}^{p^r}, \underline{x}]$ and $r \geq 1$. We can suppose that $q \in k[\underline{t}^{p^r}, \underline{x}]$ and $q' \notin k[\underline{t}^{p^{r+1}}, \underline{x}]$ for any $q' \equiv q \pmod{(h)}$. So we can say that $q(\underline{t}, \underline{x}) = f(\underline{t}^{p^r}, \underline{x})$ for some bi-homogeneous $f \in k[\underline{t}, \underline{x}]$.

Let us now consider the hypersurface $M_r \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ and the two projections $p_{M_r}: M_r \rightarrow \mathbb{P}^3$ and $g_{M_r}: M_r \rightarrow \check{\mathbb{P}}^3$. Let $V_r \subset M_r$ be the hypersurface determined by $f = 0$. Since the morphism $F_{M_r}: M \rightarrow M_r$ in diagram (8) is the p^r -th power on the $\{t_i\}$, we see that V is the following fibred product:

$$(10) \quad \begin{array}{ccc} V & \xrightarrow{F_r} & V_r \\ j \downarrow & & \downarrow i \\ M & \xrightarrow{F_{M_r}} & M_r. \end{array}$$

So, being V integral, V_r must be integral too.

π_r is dominant because the composition $V \rightarrow V_r \rightarrow \mathbb{P}^3$ is dominant. Now we show that π_r is generically smooth. $V_r \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ is determined by the complete intersection $\sum_{i=0}^3 t_i x_i^{p^r} = f(\underline{t}, \underline{x}) = 0$. If π_r were not generically smooth, then by Proposition 3.2 we could suppose that $f \in k[\underline{t}^p, \underline{x}]$. This would imply, by the commutative diagram (10), that we could take $q \in k[\underline{t}^{p^{r+1}}, \underline{x}]$, which contradicts the choice of r . \blacksquare

Let us now return to Lemma 3.1.

Proof of Lemma 3.1. From (7) we have:

$$\left(\sum_{i=0}^3 t_i x_i^{p^r} \right) \frac{\partial q}{\partial t_j} - x_j^{p^r} \left(\sum_{i=0}^3 t_i \frac{\partial q}{\partial t_i} \right) \in (h_r) \quad \forall j.$$

Using that $\sum t_i x_i^{p^r} = h_r$ and that h_r is irreducible, we deduce $\sum t_i \partial q / \partial t_i \in (h_r)$. However $\sum_{i=0}^3 t_i \frac{\partial q}{\partial t_i} = aq$, where a is the remainder of the division of α by p , because q is homogeneous of degree α in the $\{t_i\}$. So $aq \in (h_r)$ and by hypothesis the only possibility is that $a = 0$, which means that $p \mid \alpha$.

By (7) for every $i, j = 0, 1, 2, 3$ there exists l_{ij} bi-homogeneous in $k[\underline{t}, \underline{x}]$ such that:

$$(11) \quad x_i^{p^r} \frac{\partial q}{\partial t_j} - x_j^{p^r} \frac{\partial q}{\partial t_i} = l_{ij} h_r.$$

The identity:

$$x_k^{p^r} \left(x_i^{p^r} \frac{\partial q}{\partial t_j} - x_j^{p^r} \frac{\partial q}{\partial t_i} \right) - x_i^{p^r} \left(x_k^{p^r} \frac{\partial q}{\partial t_j} - x_j^{p^r} \frac{\partial q}{\partial t_k} \right) + x_j^{p^r} \left(x_k^{p^r} \frac{\partial q}{\partial t_i} - x_i^{p^r} \frac{\partial q}{\partial t_k} \right) = 0$$

for every i, j, k determines the equality $x_k^{p^r} l_{ij} - x_i^{p^r} l_{kj} + x_j^{p^r} l_{ki} = 0$. So on $D_+(x_i x_j x_k)$ we have the equality:

$$\frac{l_{ij}}{(x_i x_j)^{p^r}} - \frac{l_{ki}}{(x_k x_j)^{p^r}} + \frac{l_{ki}}{(x_k x_i)^{p^r}} = 0.$$

Considered now the open covering $\mathfrak{U} = \{D_+(x_i) \mid i = 0, \dots, 3\}$ and $n = \deg l_{ij}$, we get a Čech cocycle in $\check{H}^1(\mathfrak{U}, \mathcal{O}_{\mathbb{P}^3}(n - 2p^r)) \cong H^1(\mathcal{O}_{\mathbb{P}^3}(n - 2p^r)) = 0$. So the cocycle is a coboundary and for every i, j there exist $m_i, m_j \in k[\underline{t}, \underline{x}]$ such that $l_{ij} = m_i x_j^{p^r} - m_j x_i^{p^r}$. So by (11):

$$x_i^{p^r} \left(\frac{\partial q}{\partial t_j} - m_j h_r \right) = x_j^{p^r} \left(\frac{\partial q}{\partial t_i} - m_i h_r \right) \quad \forall i, j.$$

So there exists $m \in k[\underline{t}, \underline{x}]$ such that $\partial q / \partial t_i = m_i h_r + m x_i^{p^r}$ for every i . By replacing q by $q - m h_r$ we may assume that $\partial q / \partial t_i = m_i h_r$ for every i . We want to prove that $\partial q / \partial t_i = U_i h_r^p$ for some U_i and so let us suppose that:

$$\frac{\partial q}{\partial t_i} = v_i h_r^n \quad \forall i$$

for some $n < p - 1$. Then:

$$\frac{\partial^2 q}{\partial t_i \partial t_j} = \frac{\partial v_i}{\partial t_j} h_r^n + n v_i x_j^{p^r} h_r^{n-1} \quad \forall i, j.$$

But we have also:

$$\frac{\partial^2 q}{\partial t_i \partial t_j} = \frac{\partial v_j}{\partial t_i} h_r^n + n v_j x_i^{p^r} h_r^{n-1} \quad \forall i, j.$$

So:

$$\begin{aligned} \frac{\partial v_i}{\partial t_j} h_r^n + n v_i x_j^{p^r} h_r^{n-1} &= \frac{\partial v_j}{\partial t_i} h_r^n + n v_j x_i^{p^r} h_r^{n-1} \\ \Rightarrow v_i x_j^{p^r} - v_j x_i^{p^r} &= \frac{h_r}{n} \left(\frac{\partial v_j}{\partial t_i} - \frac{\partial v_i}{\partial t_j} \right) \quad \forall i, j. \end{aligned}$$

This implies that $v_i = v x_i^{p^r} + h_r u_i$ for every i . By replacing q by $q - \frac{1}{n+1} v h_r^{n+1}$ we may assume that:

$$\frac{\partial q}{\partial t_i} = V_i h_r^{p-1} \quad \forall i.$$

We know that:

$$(12) \quad \frac{\partial^p q}{\partial t_i^p} = 0.$$

This means that:

$$\begin{aligned} \frac{\partial^{p-1}(V_i h_r^{p-1})}{\partial t_i^{p-1}} &= 0 \\ \Rightarrow \sum_{n=0}^{p-1} \binom{p-1}{n} \frac{\partial^n V_i}{\partial t_i^n} \frac{\partial^{p-1-n}(h_r^{p-1})}{\partial t_i^{p-1-n}} &= 0 \end{aligned}$$

$$\Rightarrow h_r \mid (p-1)!x_i^{p^{r+1}-p^r}V_i \quad \Rightarrow h_r \mid V_i \quad \forall i.$$

So we can suppose that:

$$\frac{\partial q}{\partial t_i} = U_i h_r^p \quad \forall i.$$

Now (12) leads us to the conclusion that:

$$\frac{\partial^{p-1} U_i}{\partial t_i^{p-1}} = 0$$

which means that in U_i , for each i , there are no terms of type t_i^{kp-1} for any $k \geq 1$. So in particular we can say that:

$$U_0 = \frac{\partial M_0}{\partial t_0}$$

for some M_0 bi-homogeneous. Now $q' = q - M_0 h_r^p$ is such that:

$$\begin{aligned} \frac{\partial q'}{\partial t_0} &= 0 \quad \text{and} \quad \frac{\partial q'}{\partial t_i} = U'_i h_r^p, \quad i = 1, 2, 3 \\ \Rightarrow \frac{\partial U'_i}{\partial t_0} &= 0, \quad i = 1, 2, 3. \end{aligned}$$

So we can find U''_1 such that:

$$\frac{\partial U''_1}{\partial t_0} = 0 \quad \text{and} \quad \frac{\partial U''_1}{\partial t_1} = U'_1.$$

If we consider $q'' = q' - U''_1 h_r^p$ we see that:

$$\frac{\partial q''}{\partial t_i} = 0, \quad i = 0, 1 \quad \text{and} \quad \frac{\partial q''}{\partial t_i} = U''_i h_r^p, \quad i = 2, 3.$$

Proceeding in this way we get $\partial q / \partial t_i = 0$ for every i . ■

4. PROOF OF THE MAIN THEOREM

Let us consider now a curve $C \subset \mathbb{P}^3$ and, following the notation of Theorem 3.3, the projections $p_{M_r}: M_r \rightarrow \mathbb{P}^3$ and $g_{M_r}: M_r \rightarrow \tilde{\mathbb{P}}^3$. Let $T_r = p_{M_r}^{-1}(C)$ and:

$$\mathcal{I}_r(m, n) = g_{M_r}^*(\mathcal{O}_{\mathbb{P}^3 \vee}(m)) \otimes_{\mathcal{O}_{M_r}} p_{M_r}^*(\mathcal{I}_C(n))$$

for every $m, n \in \mathbb{Z}$.

Proposition 4.1. *If $\mathcal{I}_r = \mathcal{I}_r(0, 0)$ and \mathcal{I}_{T_r} is the ideal sheaf of T_r in M_r , then $\mathcal{I}_r = \mathcal{I}_{T_r}$.*

Proof. First note that $\mathcal{I}_r(m, n) = \mathcal{O}_{M_r}(m, n) \otimes_{\mathcal{O}_{M_r}} p_{M_r}^*(\mathcal{I}_C)$ for any $m, n \in \mathbb{Z}$. Moreover p_M is smooth, in particular flat. So, by base change (see (8)), p_{M_r} is flat too and we can apply [8, Ch. III, Proposition 9.3] to the following commutative diagram:

$$\begin{array}{ccc} T_r & \xrightarrow{\pi} & C \\ j \downarrow & & \downarrow i \\ M_r & \xrightarrow{p_{M_r}} & \mathbb{P}^3 \end{array}$$

to get that $p_{M_r}^* i_* \mathcal{O}_C \cong j_* \pi^* \mathcal{O}_C \cong j_* \mathcal{O}_{T_r}$. This fact together with the exact sequence $0 \rightarrow p_{M_r}^* \mathcal{I}_C \rightarrow p_{M_r}^* \mathcal{O}_{\mathbb{P}^3} \rightarrow p_{M_r}^* i_* \mathcal{O}_C \rightarrow 0$, consequence of the flatness of p_{M_r} , leads us to the desired conclusion. ■

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We divide the proof in different steps.

Step 1. *There exists $S \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ integral such that the generic fibre of the projection $S \rightarrow \check{\mathbb{P}}^3$ is an integral plane curve of degree s containing Γ .*

Proof of Step 1. Let \mathcal{I}_C be the ideal sheaf of C in \mathbb{P}^3 and let $M \subset \check{\mathbb{P}}^3 \times \mathbb{P}^3$ be the incidence variety. Let us consider the two projections:

$$\begin{array}{ccc} & M & \\ p \swarrow & & \searrow g \\ \mathbb{P}^3 & & \check{\mathbb{P}}^3 \end{array}$$

and the \mathcal{O}_M -module $\mathcal{I}(m, n) = g^*(\mathcal{O}_{\mathbb{P}^3 \vee}(m)) \otimes_{\mathcal{O}_M} p^*(\mathcal{I}_C(n))$. As we have seen in Proposition 4.1 in the case $r = 0$ \mathcal{I} is the ideal sheaf of $T = p^{-1}(C)$ in \mathcal{O}_M . Moreover there exists α such that $H^0(\mathcal{I}(\alpha, s)) \neq 0$. Indeed, if $\eta \in \check{\mathbb{P}}^3$ denotes the generic point and Γ is the generic plane section of C , then $H^0(p^*\mathcal{I}(s)|_{g^{-1}(\eta)}) = H^0(\mathcal{I}_\Gamma(s)) \neq 0$ and so this global section determines an effective divisor in $M_{k(\eta)} = M \times_{\check{\mathbb{P}}^3} \text{Spec } k(\eta) \cong \mathbb{P}_{k(\eta)}^2$. Then there exists $U \subset \check{\mathbb{P}}^3$ such that this divisor extends to an effective divisor $D \subset M_U = M \times_{\check{\mathbb{P}}^3} U$ containing $T \times_{\check{\mathbb{P}}^3} U$. The closure $\overline{D} \subset M$ is an effective divisor containing T and, since $\text{Pic}(M) = \mathbb{Z} \times \mathbb{Z}$, it is a divisor determined by a global section of $\mathcal{I}(\alpha, s)$, for some $\alpha \geq 0$.

Taken the least α such that $h^0(\mathcal{I}(\alpha, s)) \neq 0$, there exists $q \in H^0(\mathcal{I}(\alpha, s))$ that determines a hypersurface S in M such that $S \cap g^{-1}(\eta)$ is an integral curve of degree s containing Γ . Moreover, as we saw in Proposition 3.2, S is a complete intersection of codimension 2 in $\check{\mathbb{P}}^3 \times \mathbb{P}^3$. This implies that S is irreducible. \square

To prove Theorem 1.2, we now assume that the curve C is not contained in any surface of degree s , in other words, $h^0(\mathcal{I}_C(s)) = 0$. Then $p_S: S \rightarrow \mathbb{P}^3$ is dominant and, since $\alpha \geq 0$, in such a situation it must be $\alpha > 0$.

Step 2. *We can factor p_S through a generically smooth morphism $S_r \rightarrow \mathbb{P}^3$, with S_r scheme of zeroes of a global section of $\mathcal{I}_r(\beta, s)$, being $\mathcal{I}_r = p_{M_r}^*\mathcal{I}_C$, and $\alpha = \beta p^r$, for some $r \geq 0$.*

Proof of Step 2. If p_S is not generically smooth, then by Theorem 3.3 it follows that there exist $r \geq 1$, and $S_r \subset M_r$ integral hypersurface such that p_S can be factored in the following way:

$$\begin{array}{ccc} S & \xrightarrow{p_S} & \mathbb{P}^3 \\ & \searrow F^r & \nearrow p_{S_r} \\ & S_r & \end{array}$$

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where the projection p_{S_r} is dominant and generically smooth and F^r is induced by the commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{F^r} & S_r \\ j \downarrow & & \downarrow i \\ M & \xrightarrow{F_{M_r}} & M_r. \end{array}$$

Moreover, we also get that $\alpha = p^r \beta$, for some $\beta \in \mathbb{N}$, $\beta > 0$.

Considered the sheaf $\mathcal{I}_r = p_{M_r}^* \mathcal{I}_C$ and the scheme $T_r = p_{M_r}^{-1}(C)$, by Proposition 4.1 we see that \mathcal{I}_r is the ideal sheaf of T_r in M_r . Given $T = p^{-1}(C)$, since $S \supset T$ and $F_{M_r}(T) = T_r$, then $S_r \supset T_r$. So $S_r \subset M_r$ is the scheme of zeros of a global section in $H^0(\mathcal{I}_r(\beta, s))$.

Hence in both cases we find S_r integral hypersurface in M_r , with $r \geq 0$, such that the projection $p_{S_r}: S_r \rightarrow \mathbb{P}^3$ is generically smooth and $S_r \subset M_r$ is the scheme of zeros of a global section in $H^0(\mathcal{I}_r(\beta, s))$, for some $\beta > 0$. \square

Let us now follow the proof of Gruson and Peskine given in [6].

Step 3. *There exists a 3-dimensional scheme Y , with $T_r \subseteq Y \subset S_r$, such that we have:*

$$(13) \quad 0 \rightarrow \Omega_{S_r/\mathbb{P}^3}^\vee \rightarrow \Omega_{M_r/\mathbb{P}^3}^\vee \otimes_{\mathcal{O}_{M_r}} \mathcal{O}_{S_r} \rightarrow \mathcal{I}_Y(\beta, s) \rightarrow 0$$

with $\mathcal{I}_Y \subset \mathcal{O}_{S_r}$ ideal sheaf of Y .

Proof of Step 3. Since S_r is generically smooth over \mathbb{P}^3 , we have the exact sequence $0 \rightarrow \mathcal{O}_{S_r}(-\beta, -s) \rightarrow \Omega_{M_r/\mathbb{P}^3} \otimes_{\mathcal{O}_{M_r}} \mathcal{O}_{S_r} \rightarrow \Omega_{S_r/\mathbb{P}^3} \rightarrow 0$. Dualizing with respect to \mathcal{O}_{S_r} , we get:

$$(14) \quad 0 \rightarrow \Omega_{S_r/\mathbb{P}^3}^\vee \rightarrow \Omega_{M_r/\mathbb{P}^3}^\vee \otimes_{\mathcal{O}_{M_r}} \mathcal{O}_{S_r} \rightarrow \mathcal{O}_{S_r}(\beta, s).$$

Since all the fibres of the projection $T_r \rightarrow C$ have dimension 2 and $\dim S_r = 4$, p_{S_r} is not regular in any of the points of T_r . It means that the last map in (14) has image inside $\mathcal{I}_{T_r}(\beta, s)$, the ideal sheaf of T_r in S_r . So this image is an ideal sheaf of type $\mathcal{I}_Y(\beta, s)$, where $Y \subset S_r$ is a scheme containing T_r , and $3 = \dim T_r \leq \dim Y \leq \dim S_r = 4$. Since S_r is reduced and irreducible, if $\dim Y = 4$, then p_{S_r} would be non regular almost everywhere in S_r . This contradicts the fact that p_{S_r} is generically smooth. So $\dim Y = \dim T_r = 3$ and $T_r \subseteq Y$. \square

Let us consider the projection $g_{M_r}: M_r \rightarrow \check{\mathbb{P}}^3$ and take any $(\underline{b}) = (\underline{d}^{p^r}) \in \check{\mathbb{P}}^3$. Then $g_{M_r}^{-1}(\underline{b}) = \{(\underline{x}, \underline{d}^{p^r}) \mid (\sum d_i x_i)^{p^r} = 0\}$. If $H = g_{M_r}^{-1}(\underline{b})_{\text{red}}$ and $D = p(g^{-1}(\underline{b})_{\text{red}})$, then there exists $U \subset \check{\mathbb{P}}^3$ open such that, taken $(\underline{b}) \in U$, D is an irreducible curve of degree s containing the plane section of C with H . Let Γ denote such a section and let $\mathcal{I}_\Gamma \subset \mathcal{O}_D$ be the its ideal sheaf.

Step 4. *If $\mathcal{M} = (\Omega_{M_r/\mathbb{P}^3})|_H$, there exist a rank two vector bundle \mathcal{N} and a zero-dimensional scheme Δ , with $\Gamma \subseteq \Delta \subset D$, such that we have:*

$$(15) \quad 0 \rightarrow \mathcal{N} \rightarrow \mathcal{M}^\vee \rightarrow \mathcal{I}_\Delta(s) \rightarrow 0$$

being $\mathcal{I}_\Delta \subset \mathcal{O}_D$ the ideal sheaf of Δ .

Proof of Step 4. Since $M = \mathbb{P}(\Theta_{\mathbb{P}^3}(-1))$, by (8) and by [3, Lemma 1.5] we see that $M_r = \mathbb{P}(F^{r*}(\Theta_{\mathbb{P}^3}(-1)))$, where we denoted by F , as in (8), the absolute Frobenius on \mathbb{P}^3 . The sheaf $\mathcal{E} = F^{r*}(\Theta_{\mathbb{P}^3}(-1))$ is determined by the exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-p^r) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \rightarrow \mathcal{E} \rightarrow 0$ and by [8, Ch. III, Ex. 8.4(b)] we have also $0 \rightarrow \Omega_{M_r/\mathbb{P}^3} \rightarrow (p_{M_r}^* \mathcal{E})(-1) \rightarrow \mathcal{O}_{M_r} \rightarrow 0$. When we restrict to H , by the fact that the sequences locally split it follows that the following sequences are exact:

$$(16) \quad 0 \rightarrow \mathcal{O}_H(-p^r) \rightarrow \mathcal{O}_H^{\oplus 4} \rightarrow \mathcal{E}_H \rightarrow 0$$

and

$$0 \rightarrow \mathcal{M} \rightarrow (p_{M_r}^* \mathcal{E})|_H(-1) \rightarrow \mathcal{O}_H \rightarrow 0$$

where $\mathcal{M} = (\Omega_{M_r/\mathbb{P}^3})|_H$. Since $(p_{M_r}^* \mathcal{E})|_H(-1) = (p_{M_r}^* \mathcal{E}(-1, 0))|_H = \mathcal{E}_H$, we have:

$$(17) \quad 0 \rightarrow \mathcal{M} \rightarrow \mathcal{E}_H \rightarrow \mathcal{O}_H \rightarrow 0.$$

Restricting (13) to H , we get a surjective map $\mathcal{M}^\vee \otimes_{\mathcal{O}_H} \mathcal{O}_D \rightarrow \mathcal{I}_\Delta(s)$, with $\mathcal{I}_\Delta \subset \mathcal{O}_D$ ideal sheaf of a zero-dimensional scheme Δ containing Γ . The kernel of this map is a locally free sheaf of rank 2 that determines the exact sequence (15). \square

Step 5. $d \leq s^2 + p^{2r}$.

Proof of Step 5. Note that $c_1(\mathcal{I}_\Delta(s)) = s$ and $c_2(\mathcal{I}_\Delta(s)) = \deg \Delta = \delta \geq d$. Now we compute the Chern classes of the other sheaves. From (16) we have $c_1(\mathcal{E}_H) = p^r$ and $c_2(\mathcal{E}_H) = p^{2r}$. So by (17) $c_1(\mathcal{M}) = p^r$ and $c_2(\mathcal{M}) = p^{2r}$, from which it follows that $c_1(\mathcal{M}^\vee) = -p^r$ and $c_2(\mathcal{M}^\vee) = p^{2r}$. By (15) we see that:

$$c_1(\mathcal{N}) = -p^r - s \quad \text{and} \quad c_2(\mathcal{N}) = p^{2r} - \delta + s^2 + p^r s.$$

Let $m \in \mathbb{Z}$ be the smallest number such that $H^0(\mathcal{M}^\vee(m-1)) = 0$ and $H^0(\mathcal{M}^\vee(m)) > 0$. Dualizing (17), since \mathcal{O}_H is a locally free sheaf, we get $0 \rightarrow \mathcal{O}_H \rightarrow \mathcal{E}_H^\vee \rightarrow \mathcal{M}^\vee \rightarrow 0$, from which it follows that:

$$h^0(\mathcal{M}^\vee(m)) = h^0(\mathcal{E}_H^\vee(m)) - h^0(\mathcal{O}_H(m)) \quad \forall m \in \mathbb{Z}.$$

From the exact sequence (16) we see that, in the notation of Lemma 2.3, $\mathcal{E}_H^\vee = \mathcal{K}$, so that, by Lemma 2.3 we see that:

$$h^0(\mathcal{E}_H^\vee(m)) = h^0(\mathcal{O}_H(m)) + 3h^0(\mathcal{O}_H(m - p^r)) - h^0(\mathcal{O}_H(m - 2p^r))$$

for every $m \in \mathbb{Z}$. So:

$$h^0(\mathcal{M}^\vee(m)) = 3h^0(\mathcal{O}_H(m - p^r)) - h^0(\mathcal{O}_H(m - 2p^r)) \quad \forall m \in \mathbb{Z}.$$

This implies that $h^0(\mathcal{M}^\vee(p^r - 1)) = 0$ and $h^0(\mathcal{M}^\vee(p^r)) > 0$. So $h^0(\mathcal{N}(p^r - 1)) = 0$ and $p^{2r} + p^r(-s - p^r) + c_2(\mathcal{N}) = c_2(\mathcal{N}(p^r)) \geq 0$ by [6]. So we get that $p^{2r} + s^2 \geq \delta$ and, since $\delta \geq d$:

$$(18) \quad p^{2r} + s^2 \geq d.$$

\square

Step 6. If C is connected, $p^r < s$; if C is merely reduced, $p^r \leq s$.

Proof of Step 6. Let us now consider a generic plane $H = V(l)$, with l linear form in the $\{x_i\}$, and the non reduced surface H_r in \mathbb{P}^3 given by $l^{p^r} = 0$. Let $\Gamma_r \subset H_r$ be the section of C with H_r . Then there is the following exact sequence:

$$0 \rightarrow \mathcal{I}_C(-p^r) \xrightarrow{\varphi_H} \mathcal{I}_C \rightarrow i_* \mathcal{I}_{\Gamma_r} \rightarrow 0$$

where $i: \Gamma_r \hookrightarrow \mathbb{P}^3$ and φ_H is the multiplication by l^{p^r} . The long cohomology exact sequence associated to the previous exact sequence shifted by s determines the following one:

$$(19) \quad H^0(\mathcal{I}_C(s)) \rightarrow H^0(\mathcal{I}_{\Gamma_r}(s)) \rightarrow H^1(\mathcal{I}_C(s-p^r)) \xrightarrow{\varphi_H} H^1(\mathcal{I}_C(s)).$$

Let $[H] \in \check{\mathbb{P}}^3$ be a point such that the fibre of the projection $M_r \rightarrow \check{\mathbb{P}}^3$ at $[H]$ is isomorphic to H_r . Then, taking $[H]$ in a suitable open $U \subset \check{\mathbb{P}}^3$, $g_{S_r}^{-1}([H])$ is the complete intersection of H_r and of a surface of degree s containing $C \cap H_r$, because $T_r \subset S_r$. It means that $H^0(\mathcal{I}_{\Gamma_r}(s)) \neq 0$ and so by (19) and by hypothesis it must be $h^1(\mathcal{I}_C(s-p^r)) \neq 0$.

Let us suppose that C is connected. Then $h^1(\mathcal{I}_C(n)) = 0$ for $n \leq 0$. So $s-p^r \geq 1$, because otherwise $h^0(\mathcal{I}_C(s)) \neq 0$, which contradicts the hypothesis made at the beginning.

If C is merely reduced, we still have $h^1(\mathcal{I}_C(n)) = 0$ for $n < 0$. So, as before, it must be $s-p^r \geq 0$. \square

Let us suppose that C is connected. By Step 6, if $p \geq s$, then the only possibility is that $r = 0$, which implies $d \leq s^2 + 1$. If $p < s$, then $p^r \leq s-1$ and, in particular, $p^r \leq p^n$, being $p^n < s \leq p^{n+1}$. So by (18) $d \leq s^2 + p^{2n}$ and in particular we see that $d \leq 2s^2 - 2s + 1$.

Let us suppose now that C is merely reduced. If $p > s$, then it must be $r = 0$, so that by (18) we have $d \leq s^2 + 1$. If $p \leq s$, then $p^r \leq s$ and, in particular, $p^r \leq p^n$, being $p^n \leq s < p^{n+1}$. Now by (18) we see that $d \leq s^2 + p^{2n}$. In particular, $d \leq 2s^2$. \blacksquare

5. EXAMPLE

In this section we show that for any p there exist smooth integral curves of degree $d = s^2 + p^{2n}$, being $s > p$ and n such that $p^n < s \leq p^{n+1}$, that are not contained in any surface of degree s and that have the generic plane section contained in an integral plane curve of degree s .

First, let us recall the following definition.

Definition 5.1. A rank 2 vector bundle \mathcal{E}_0 on \mathbb{P}^3 is said to be a *null-correlation bundle* if there exists an exact sequence:

$$(20) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tau} \Omega_{\mathbb{P}^3}(2) \rightarrow \mathcal{E}_0(1) \rightarrow 0$$

where τ is a nowhere vanishing section of $\Omega_{\mathbb{P}^3}(2)$.

Remark 5.2. It is possible to prove (see [1], [14] and [9, Example 8.4.1]) that \mathcal{E} is a stable rank 2 vector bundle on \mathbb{P}^3 with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 1$ if and only if \mathcal{E} is isomorphic to a null-correlation bundle.

Example 5.3. Let \mathcal{E}_0 be a null-correlation bundle. Let $n, s \in \mathbb{N}$ be positive integers and let $F: \mathbb{P}^3 \rightarrow \mathbb{P}^3$ be the absolute Frobenius on \mathbb{P}^3 . Let us consider the sheaf $\mathcal{E}(s) = F^{n*}(\mathcal{E}_0) \otimes \mathcal{O}_{\mathbb{P}^3}(s)$. Since $c_1(F^{n*}(\mathcal{E}_0)) = 0$ and $c_2(F^{n*}(\mathcal{E}_0)) = p^{2n}$, we see that $c_1(\mathcal{E}(s)) = 2s$ and $c_2(\mathcal{E}(s)) = p^{2n} + s^2$.

Let $\sigma \in H^0(\mathcal{E}(s))$ be a global section such that the zero locus of σ is a curve C . Then we get the exact sequence:

$$(21) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{E}(s) \rightarrow \mathcal{I}_C(2s) \rightarrow 0$$

so that $h^0(\mathcal{I}_C(s)) = h^0(\mathcal{E})$ and $\deg C = c_2(\mathcal{E}(s)) = p^{2n} + s^2$. Let H be a plane transversal to C and $\Gamma = C \cap H$. Restricting to H the exact sequence (21) we have:

$$(22) \quad 0 \rightarrow \mathcal{O}_H \rightarrow \mathcal{E}(s)|_H \rightarrow \mathcal{I}_\Gamma(2s) \rightarrow 0$$

so that:

$$(23) \quad h^0(\mathcal{I}_\Gamma(s)) = h^0(\mathcal{E}|_H).$$

By [3, Theorem 3.2] \mathcal{E} is stable and we can choose H sufficiently general in such a way that $\mathcal{E}|_H$ is semi-stable, but not stable. Since \mathcal{E} is stable and $c_1(\mathcal{E}) = 0$, then by [10, Lemma 3.1] $h^0(\mathcal{E}) = 0$, which implies that $h^0(\mathcal{I}_C(s)) = 0$. So C is not contained in any surface of degree s . Since $\mathcal{E}|_H$ is semi-stable, but not stable and $c_1(\mathcal{E}|_H) = 0$, it must be $h^0(\mathcal{E}|_H) \neq 0$, so that $h^0(\mathcal{I}_\Gamma(s)) \neq 0$. Moreover by (22) $h^0(\mathcal{I}_\Gamma(s-1)) = h^0(\mathcal{E}|_H(-1))$. Now note that by (20) the sheaf \mathcal{E} is determined by the exact sequence:

$$(24) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-p^n) \rightarrow (F^n)^*(\Omega_{\mathbb{P}^3})(p^n) \rightarrow \mathcal{E} \rightarrow 0.$$

so that, considered the sheaf $\mathcal{F} = (F^n)^*(\Omega_{\mathbb{P}^3})$, we have the exact sequence $0 \rightarrow \mathcal{O}_H(-p^n-1) \rightarrow \mathcal{F}|_H(p^n-1) \rightarrow \mathcal{E}|_H(-1) \rightarrow 0$, which implies that $h^0(\mathcal{E}|_H(-1)) = h^0(\mathcal{F}|_H(p^n-1)) = 0$ by Lemma 2.3. So the plane curves of degree s containing the generic plane section of C are the minimal ones. Moreover by the previous exact sequence and by Lemma 2.3 we have the equality $h^0(\mathcal{E}|_H) = h^0(\mathcal{F}|_H(p^n)) = 1$, which implies by (23) that $h^0(\mathcal{I}_\Gamma(s)) = 1$. So there is a unique plane curve of degree s containing Γ , which means that this plane curve of degree s is the minimal plane curve containing Γ .

Now we want to know when $h^0(\mathcal{E}(s)) \neq 0$. By (24) we get for each $s \in \mathbb{N}$:

$$(25) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-p^n+s) \rightarrow \mathcal{F}(p^n+s) \rightarrow \mathcal{E}(s) \rightarrow 0.$$

By Lemma 2.2 $h^0(\mathcal{F}(2p^n)) = 6$ and $h^0(\mathcal{F}(p^n+s)) \neq 0$ if and only if $s \geq p^n$. So from (25):

$$(26) \quad h^0(\mathcal{E}(p^n)) = 5$$

and $h^0(\mathcal{E}(s)) \neq 0$ if and only if $s \geq p^n$.

So we have global sections only for $s \geq p^n$. We want to prove there exist global sections of $\mathcal{E}(s)$, for every $s \geq p^n$, whose zero locus is a curve. First we must prove that the \mathcal{E} is not split. If this was the case, then, being \mathcal{E} a locally free sheaf of rank 2, we would have $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b)$, for some $a, b \in \mathbb{Z}$. Since $h^0(\mathcal{E}(p^n-1)) = 0$, it must be $a + p^n - 1 < 0$ and $b + p^n - 1 < 0$, so that $h^0(\mathcal{E}(p^n)) \leq 2$, but this contradicts (26). So \mathcal{E} is not split. Moreover, since $h^0(\mathcal{E}(p^n)) = 5 > h^0(\mathcal{O}_{\mathbb{P}^3}) = 1$, by [5, Theorem 0.1] we get that every general nonzero global section of $\mathcal{E}(s)$, for $s \geq p^n$, has as zero locus a curve in \mathbb{P}^3 .

Now we want to know when there are connected curves. By [9, Proposition 1.4] if $h^1(\mathcal{E}^\vee(-s)) = 0$, then a generic global section of $\mathcal{E}(s)$, for $s \geq p^n$, determines a connected curve. Note that $h^1(\mathcal{E}^\vee(-s)) = h^2(\mathcal{E}(s-4))$. By (25) and by Lemma 2.2 $h^2(\mathcal{E}(s-4)) \leq h^3(\mathcal{O}_{\mathbb{P}^3}(s-p^n-4)) = 0$ for $s > p^n$. So for $s > p^n$ the generic global section of $\mathcal{E}(s)$ is connected.

Now we want to know when we have nonsingular curves. By [9, Proposition 1.4], if $\mathcal{E}(s-1)$ is generated by its global sections, then a sufficiently generic global section in $H^0(\mathcal{E}(s))$ will determine a nonsingular zero locus (not necessarily connected). Note now that in the proof of Lemma 2.2 we have seen that there is a surjective morphism of sheaves $\mathcal{O}_{\mathbb{P}^3}^{\oplus 6} \rightarrow \mathcal{F}(2p^n)$ (see (4)). From (25) we see that we have also the surjective morphism $\mathcal{F}(2p^n) \rightarrow \mathcal{E}(p^n)$. So we get a surjective morphism $\mathcal{O}_{\mathbb{P}^3}^{\oplus 6} \rightarrow \mathcal{E}(p^n)$, which means that $\mathcal{E}(p^n)$ is generated by its global sections and so $\mathcal{E}(s)$ is generated by its global sections for $s \geq p^n$.

In this way we construct, for any p, n, s , with $s \geq p^n$, examples of curves $C \subset \mathbb{P}^3$ of degree $p^{2n} + s^2$ not contained in any surface of degree s such that the minimal curve containing its generic plane section has degree s and such that:

- (1) C is nonsingular, in particular reduced;
- (2) C is nonsingular and connected, which means nonsingular and irreducible, in the case $s > p^n$. In this situation the minimal curve of degree s containing the generic plane section of C is integral by [2, Theorem 4.1].

In particular, we see that the bound in Theorem 1.2 for connected curves is sharp. Moreover, taking $s = p^n + 1$, we see that there exist connected and reduced curves (in particular nonsingular) of degree $d = 2s^2 - 2s + 1$, not lying on any surface of degree s , whose generic plane section is contained in an integral plane curve of degree s .

REFERENCES

- [1] W. P. Barth, *Some properties of stable rank-2 vector bundles on \mathbb{P}^n* , Math. Ann. **226** no. 2 (1977), 125–150.
- [2] P. Bonacini, *On the plane section of an integral curve in positive characteristic*, to appear in Proc. A.M.S.
- [3] L. Ein, *Stable vector bundles on projective spaces in char $p > 0$* , Math. Ann. **254** (1980), 53–72.
- [4] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Graduate Texts in Mathematics, no. 150, Springer-Verlag, New York, 1995.
- [5] A. Geramita, M. Roggero, P. Valabrega, *Subcanonical curves with the same postulation as Q skew complete intersections in projective 3-space*, Istit. Lombardo Accad. Sci. Lett. Rend. A **123** (1989), 111–121 (1990).
- [6] L. Gruson, C. Peskine, *Section plane d’une courbe gauche: postulation*, Enumerative Geometry and Classical Algebraic Geometry (Nice, 1981), Progress in Math., no. 24, Birkhäuser, Boston, Mass., 1982, pp. 33–35.
- [7] J. Harris, *The Genus of space curves*, Math. Ann. **249** (1980), 191–204.
- [8] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, no. 52, Springer Verlag, New York, 1977.
- [9] ———, *Stable vector bundles of rank 2 on \mathbb{P}^3* , Math. Ann., **238** 3 (1978), 229–280.
- [10] ———, *Stable reflexive sheaves*, Math. Ann., **254** 2 (1980), 121–176.
- [11] O. A. Laudal, *A generalized trisecant lemma*, Algebraic Geometry (Proc. Sympos. Univ. Tromsø, Tromsø, 1977), Lecture Notes in Math., no. 687, Springer-Verlag, Berlin, 1978, pp. 112–149.
- [12] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, no. 33, Princeton University Press, Princeton, N.J., 1980.
- [13] R. Strano, *On generalized Laudal’s Lemma*, Complex projective geometry (Trieste 1989/Bergen 1989), London Math. Soc. Lecture Note Ser., no. 179, Cambridge Univ. Press, Cambridge, 1992, pp. 284–293.
- [14] G. P. Wever, *The moduli of a class of rank 2 vector bundles on \mathbb{P}^n* , Nagoya Math. J., **84** (1981), 9–30.

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